

UNIFORM KAZHDAN GROUPS

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ABSTRACT. We construct first examples of infinite groups having property (T) whose Kazhdan constants admit a lower bound independent of the choice of a finite generating set.

1. INTRODUCTION

Let G be a group generated by a finite set X . We say that G has property (U) (with respect to X) if there is a constant $C = C(G)$ such that the following condition holds. For any generating set Y of G , there exists an element $t \in G$ such that

$$\max_{x \in X} |t^{-1}xt|_Y \leq C.$$

Observe that if G has property (U) with respect to X , it has (U) with respect to any other finite generating set. Finite groups are obvious examples of groups with property (U). On the other hand, the question of the existence of infinite groups with this property is not trivial. In this paper we prove the following.

Theorem 1.1. *Every non-elementary torsion-free hyperbolic group has an infinite quotient group with property (U).*

Our construction of groups with property (U) was inspired by a well-known question about Kazhdan groups. We recall that to each finite subset S of a (discrete) group G and each unitary representation $\pi: G \rightarrow U(\mathcal{H})$ of G on a Hilbert space \mathcal{H} , one associates the *Kazhdan constant*

$$\kappa(G, S, \pi) = \inf_{\xi \in \mathcal{H}^1} \max_{s \in S} \|\pi(s)\xi - \xi\|,$$

where \mathcal{H}^1 denotes the unit sphere in \mathcal{H} . Further we set $\kappa(G, S) = \inf_{\pi} \kappa(G, S, \pi)$ and $\kappa(G) = \inf_S \kappa(G, S)$, where the infimum is taken over all unitary representations π and all finite generating sets S of G respectively. Recall that a group G has property (T) of Kazhdan if $\kappa(G, S) > 0$ for some (or, equivalently, for any) generating set S of G .

Clearly $\kappa(G) > 0$ whenever G is a finite group. In [6], Lubotzky asked whether this inequality holds for all Kazhdan groups G . The negative answer was obtained by Gelander and Żuk [2]. They proved that any dense subgroup of a connected locally compact topological group has zero uniform Kazhdan constant. Shortly later Osin showed that $\kappa(H) = 0$ for any infinite hyperbolic group H . Thus the equality $\kappa(H) = 0$ holds for the majority of known groups with property (T). Moreover, no examples of infinite groups with non-zero uniform Kazhdan constant were known until now.

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It is easy to show that every group having properties (U) and (T) has non-zero uniform Kazhdan constant. Applying Theorem 1.1 to a torsion-free Kazhdan hyperbolic group, we obtain the following.

Corollary 1.2. *There exists an infinite finitely generated group G such that $\kappa(G) > 0$.*

The proof of Theorem 1.1 is based on a variant of Ol'shanskii's technique [8] as elaborated by Semenov [12]. It is worth to note that our group G satisfies the identity $x^n = 1$ for some large odd n . In particular, G is not residually finite according to the positive solution of the restricted Burnside problem [14]. Our method does not allow to avoid this identity. Thus the following question is still open.

Question 1.3. *Does there exist an infinite residually finite group G such that $\kappa(G)$ is positive?*

2. THE OL'SHANSKII–SEME NOV CONSTRUCTION

Let us give one of many equivalent definitions of a hyperbolic group [3]. A group G with a finite generating set X is hyperbolic (in the sense of Gromov) if its Cayley graph $\Gamma = \Gamma(G, X)$ is a hyperbolic metric space with respect to the word-length metric. This means that there exists a constant δ such that every geodesic triangle in Γ is δ -thin, i.e., each of its sides belongs to the closed δ -neighborhood of the union of the other two sides. A group is called elementary if it contains a cyclic subgroup of finite index.

In [8] Ol'shanskii showed that any non-elementary torsion free hyperbolic group has an infinite quotient satisfying the identity $x^n = 1$ provided n is large and odd (later Ivanov and Ol'shanskii [4] showed how to construct such quotients of hyperbolic groups with torsion). The proof was based on the graded diagrams method [9]. This method was also used in [7], where Ol'shanskii constructed infinite finitely generated groups all of whose proper subgroups are finite cyclic. In [12, 13], Semenov incorporated the two constructions to obtain infinite quotients of non-elementary torsion free hyperbolic groups all of whose proper subgroups are finite cyclic.

In this paper we use the methods of [9] as elaborated in [12, 13]. Our proofs heavily depend on technical lemmas from [12] that are collected below (otherwise our paper would be unreasonably long). Let us recall the main steps of the Ol'shanskii–Semenov construction.

Let

$$G = \langle a_1, a_2, \dots, a_m \mid R = 1, R \in \mathcal{R}_0 \rangle$$

be a non-elementary torsion free hyperbolic group. In what follows, certain parameters (C, d, h, n, n_0) appear. In fact, we do not need the exact values of these parameters. For our goals it suffices to know that *there exist* parameters C, d, h, n, n_0 such that all results listed below are true and, in addition, $n_0 \gg n \gg 1$ and n_0 is an odd number. (For exact values of these parameters and other details we refer to [8, 12, 13].)

We construct a sequence of quotient group $G(i)$ of G as follows. Let $G(0) = G$. Assuming that the group $G(i-1) = \langle a_1, a_2, \dots, a_m \mid R = 1, R \in \mathcal{R}_{i-1} \rangle$ is already constructed, we will introduce the set \mathcal{S}_i of defining words of rank i , and set $\mathcal{R}_i = \mathcal{R}_{i-1} \cup \mathcal{S}_i$ and

$$G(i) = \langle a_1, a_2, \dots, a_m \mid R = 1, R \in \mathcal{R}_i \rangle$$

Elements of $G(i)$ are referred to as words over the alphabet $\{a_1^{\pm 1}, \dots, a_m^{\pm 1}\}$. By $|W|$ we denote the length of a word W with respect to $\{a_1^{\pm 1}, \dots, a_m^{\pm 1}\}$. We write $A \equiv B$ to indicate letter-for-letter equality of words A and B .

We say that a word W is simple in $G(0)$ and in all ranks $i < C$ if it is not equal to a proper power of some element of $G(0)$ and is not conjugate to any word of smaller length. A word W is called simple in rank $i \geq C$ if it is not conjugated to a power of a shorter word in the group $G(i-1)$ and is not conjugated to a power of a period of rank $k \leq i-1$ in the group $G(i-1)$. For each $i = 1, 2, \dots$ choose some set \mathcal{X}_i of simple in rank $i-1$ words of length i , maximal with respect to the property that if $A, B \in \mathcal{X}_i$, $A \neq B$, then A is not conjugated to $B^{\pm 1}$ in the group $G(i-1)$. Words from \mathcal{X}_i are called periods of rank i . For every period $A \in \mathcal{X}_i$ choose a maximal subset of words \mathcal{G}_A such that:

- 1) if $T \in \mathcal{G}_A$, then $0 < |T| < d|A|$;
- 2) every double coset of the group $G(i-1)$ over the pair of subgroups $\langle A \rangle, \langle A \rangle$ contains at most one word from \mathcal{G}_A , and this word has minimal length among words from the coset.

The set of defining words \mathcal{S}_i of rank i is constructed as follows. First, include in \mathcal{S}_i all words A^{n_0} for each $A \in \mathcal{X}_i$. Furthermore, for every $A \in \mathcal{X}_i$, if $a_j \notin \langle A \rangle \subset G(i-1)$, and $a_j \notin \langle A \rangle a_k \langle A \rangle$ for $k < j$, then for every $T \in \mathcal{G}_A \setminus (\langle A \rangle a_j \langle A \rangle)$ we include in \mathcal{S}_i the word

$$a_j A^{n+j-1} T A^{n+m+j-1} \dots T A^{n+m(h-1)+j-1},$$

where j runs from 1 to m .

Then set $\mathcal{R}_i = \mathcal{R}_{i-1} \cup \mathcal{S}_i$, define $G(i) = \langle a_1, a_2, \dots, a_m | R = 1, R \in \mathcal{R}_i \rangle$ and finally,

$$G(\infty) = \langle a_1, a_2, \dots, a_m | R = 1, R \in \mathcal{R} = \bigcup_{i=0}^{\infty} \mathcal{R}_i \rangle$$

The following lemmas can be found in [12] (see also [9, Chapter 8, 9])

Lemma 2.1. *The group $G(\infty)$ is infinite. Every proper subgroup of $G(\infty)$ is cyclic of order dividing n_0 .*

Lemma 2.2. *Any nontrivial element of $G(\infty)$ is conjugate to a power of a period of certain rank. The centralizer of any nontrivial element of $G(\infty)$ is cyclic of order n_0 .*

For the proof of the following lemma we refer to [10], Corollary 2.3 :

Lemma 2.3. *If $a, b \in G(\infty)$, $[a, b] \neq 1$, then for every $i \in \mathbb{Z}$, we have $[a^i, b] \neq 1$ whenever $a^i \neq 1$.*

In the next two lemmas additional parameter σ appears. It corresponds to $100k_1^{-3/2}\zeta^{-1}$ from [12].

Lemma 2.4. *Let C be a period of a certain rank, $V \equiv C^k$, where $\sigma < k < n_0/2$. Suppose that an element W does not commute with V and has minimal length among the elements of $\langle C^k \rangle W \langle C^k \rangle$. Assume also that $[C^k, W]$ is conjugated to A^l , where A is a period of a certain rank and $|l| \leq n_0/2$. Then $|l| \leq \sigma$ and the pair $([C^k, W], C^k)$ is conjugate to the pair (A^l, B) , where $|B| < d|A|$.*

Lemma 2.5. *Let l be an integer and A, B be elements of $G(\infty)$ such that*

- (1) A is a period of certain rank;
- (2) $[A, B] \neq 1$ in $G(\infty)$;
- (3) $|l| \leq \sigma$ and $|B| < d|A|$.

Then there exists an integer s such that the pair (BA^{ls}, B) is conjugate to a pair (F, T) , where F is a period of certain rank, $[F, T] \neq 1$ and $|T| < d|F|$.

3. PROOF OF THE MAIN RESULTS

Proof of Theorem 1.1. We show that the group $G(\infty)$ construction of which was outlined in Section 2 has property (U) with respect to the generating set $\{a_1, a_2, \dots, a_m\}$. In view of Lemma 2.1 it suffices to consider generating sets Y of $G(\infty)$ consisting of two non-commuting elements. First we will consider the situation when Y consists of a period of some rank and a "short" word. Then it will be shown how to reduce general situation to the considered one.

Assuming that generating set Y consists of a period F and a word T such that $[F, T] \neq 1$ and $|T| < d|F|$, we show that there is a uniform bound (depending on the group $G(\infty)$ only) for the lengths of elements a_1, \dots, a_m as words in F and T . Denote by i the rank of period F and consider the relations imposed on the i -th step.

If $a_j \in \langle F \rangle \subset G(i-1)$ or $T \in \langle F \rangle a_j \langle F \rangle \subset G(i-1)$, then, in the group $G(\infty)$, $|a_j|_{\{F, T\}} < n_0$ in view of the relation $F^{n_0} = 1$. Suppose now that $a_j \notin \langle F \rangle \cup (\langle F \rangle T \langle F \rangle) \subset G(i-1)$. If l is minimal index such that $a_l \in \langle F \rangle a_j \langle F \rangle$, then for some element $T_1 \in \langle F \rangle T \langle F \rangle$ we encounter a relation

$$a_l F^{n+j-1} T_1 F^{n+m+j-1} \dots T_1 F^{n+m(h-1)+j-1} = 1.$$

Again, in view of $F^{n_0} = 1$, one has $|T_1|_{\{F, T\}} < n_0$ and therefore $|a_l|_{\{F, T\}} < n_0(n_0 + m)$ in the group $G(\infty)$. Consequently, $|a_j|_{\{F, T\}} < n_0(n_0 + m + 1)$ for any $a_j \in \langle F \rangle a_l \langle F \rangle$ in $G(\infty)$. Thus, every a_j is equal to a word in F and T of length at most $n_0(n_0 + m + 1)$ in the group $G(\infty)$.

Let now Y consist of two non-commuting elements u and v . We will show that there exist a constant M depending on the group $G(\infty)$ only such that the following condition holds. There exists a period F and a word T , $[F, T] \neq 1$, $|T| < d|F|$, such that

$$\max(|t^{-1} F t|_{\{u, v\}}, |t^{-1} T t|_{\{u, v\}}) < M$$

for some element $t \in G(\infty)$.

By Lemma 2.2, we have $u = P^{-1} C^{k'} P$ for some element P and a period C of certain rank. Without loss of generality we can assume that $0 < k' < n_0/2$. Recall that $\sigma < n_0/4$. There is a number $i \in \mathbb{N}$ such that

$$i < \frac{n_0}{2}$$

and $\sigma < i k' < n_0/2$. Thus we have

$$u^i = P^{-1} C^k P,$$

where

$$\sigma < k < \frac{n_0}{2}.$$

Using Lemma 2.3, we note that

$$[C^k, P v P^{-1}] = P[u^i, v] P^{-1} \neq 1.$$

Denote by W_0 the element $P v P^{-1}$ and by W the shortest element in the double coset $\langle C^k \rangle W_0 \langle C^k \rangle$. There are some integers k_1, k_2 satisfying the inequality $\max(|k_1|, |k_2|) <$

$n_0/2$, such that $W = C^{kk_1}W_0C^{kk_2}$. Since $W = Pu^{ik_1}vu^{ik_2}P^{-1}$, we obtain the following inequality:

$$|P^{-1}WP|_{\{u, v\}} \leq 2i + k_1 + k_2 + 1 < 2n_0.$$

The elements C^k and W do not commute with each other. Therefore, by Lemma 2.2, $[C^k, W]$ is conjugate to A^l , where A is a period of a certain rank:

$$[C^k, W] = Q^{-1}A^lQ$$

for some element Q . Set

$$B = QC^kQ^{-1}.$$

Applying Lemma 2.4, we obtain $|l| \leq \sigma$ and $|B| < d|A|$. Note that all conditions of Lemma 2.5 are satisfied for A and B . Therefore, there is an integer s such that the pair (BA^{ls}, B) is conjugate to a pair (F, T) , where F is a period of a certain rank, $[F, T] \neq 1$, and $|T| < d|F|$. We can assume that $s < n_0$, so that

$$\max(|Q^{-1}BA^{ls}Q|_{\{[C^k, W], C^k\}}, |Q^{-1}BQ|_{\{[C^k, W], C^k\}}) < n_0 + 1.$$

Note that

$$\max(|P^{-1}[C^k, W]P|_{\{u, v\}}, |P^{-1}C^kP|_{\{u, v\}}) < 5n_0.$$

Therefore for some element t ,

$$\max(|t^{-1}Ft|_{\{u, v\}}, |t^{-1}Tt|_{\{u, v\}}) < 5n_0(n_0 + 1).$$

To complete the proof it suffices to set $M = 5n_0(n_0 + 1)$ and $C = C(G(\infty)) = Mn_0(n_0 + m + 1)$. \square

To prove Corollary 1.2 we need two elementary facts about Kazhdan constants.

Lemma 3.1. *Let G be a group, X a finite generating set of G . Then:*

- (1) $\varkappa(G, X) = \varkappa(G, t^{-1}Xt)$ for any element $t \in G$.
- (2) For any finite generating set Y in G , $\varkappa(G, Y) \geq \varkappa(G, X)/d$, where $d = \max_{x \in X} |x|_Y$.

Proof. For any $g, t \in G$, any unitary presentation $\pi : G \rightarrow U(\mathcal{H})$, and any unit vector $\xi \in \mathcal{H}$, we have

$$\|\pi(t^{-1}gt)\xi - \xi\| = \|\pi(gt)\xi - \pi(t)\xi\| = \|\pi(g)\xi - \xi\|.$$

This implies the first assertion of the lemma.

Further let $x = y_1y_2 \cdots y_l$ for some $y_1, \dots, y_l \in Y \cup Y^{-1}$ and let $\xi \in \mathcal{H}^1$. Then

$$\|\pi(x)\xi - \xi\| \leq \|\pi(y_1)\xi - \xi\| + \sum_{i=1}^{l-1} \|\pi(y_1y_2 \cdots y_{i+1})\xi - \pi(y_1y_2 \cdots y_i)\xi\| =$$

$$\sum_{i=1}^l \|\pi(y_i)\xi - \xi\| \leq l \max_{y \in Y} \|\pi(y)\xi - \xi\|.$$

This yields the second assertion. \square

Proof of Corollary 1.2. Applying Theorem 1.1 to a torsion free hyperbolic Kazhdan group, one obtains group $G(\infty)$ satisfying both properties (T) and (U). We denote by $\varkappa = \varkappa(G(\infty), X)$ the Kazhdan constant of $G(\infty)$ with respect to the generating set $X = \{a_1, a_2, \dots, a_m\}$. If Y is any other generating set of $G(\infty)$, then, by Lemma 3.1,

$$\varkappa(G(\infty), Y) \geq \frac{1}{C} \varkappa(G(\infty), X),$$

where $C = C(G(\infty))$ is the constant realizing property (U) of $G(\infty)$ with respect to X . Consequently, property (T) of $G(\infty)$ implies $\kappa(G(\infty)) > 0$. \square

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